

Maxwell-Dirac stress-energy tensor in terms of Fierz bilinear currents

S M Inglis and P D Jarvis

School of Mathematics and Physics, University of Tasmania, Sandy Bay Campus, Private Bag 37, Hobart, Tasmania, 700

Email: sminglis@utas.edu.au, Peter.Jarvis@utas.edu.au

Abstract. We analyze the stress-energy tensor for the self-coupled Maxwell-Dirac system in the bilinear current formalism, using two independent approaches. The first method used is that attributed to Belinfante: starting from the spinor form of the action, the well-known canonical stress-energy tensor is augmented, by extending the Noether symmetry current to include contributions from the Lorentz group, to a manifestly symmetric form. This form admits a transcription to bilinear current form. The second method used is the variational derivation based on the covariant coupling to general relativity. The starting point here at the outset is the transcription of the action using, as independent field variables, both the bilinear currents, together with a gauge invariant vector field (a proxy for the electromagnetic vector potential). A central feature of the two constructions is that they both involve the mapping of the Dirac contribution to the stress-energy from the spinor fields to the equivalent set of bilinear tensor currents, through the use of appropriate Fierz identities. Although this mapping is done at quite different stages, nonetheless we find that the two forms of the bilinear stress-energy tensor agree. Finally, as an application, we consider the reduction of the obtained stress-energy tensor in bilinear form, under the assumption of spherical symmetry.

1 Introduction

In classical field theory, it is well known that a conserved energy-momentum tensor for a physical system can be constructed by the Noether method, exploiting invariance of the Lagrangian density under space-time translations. However, the derived symmetry current using this method, the so-called “canonical” form [10], has the unfortunate drawbacks of being neither symmetric in its two tensor indices, nor gauge invariant.

Many attempts have been made to rectify these problems, two of the most prominent being the Belinfante form [1], and the variational form from general relativity [22]. The basis of the Belinfante approach is to extend the invariance of the Lagrangian to include contributions from Lorentz transformations, so that the Noether symmetry current becomes that associated with the full Poincaré group. In this way, the canonical term is symmetrized [23], and extra “correction” terms are present, which are attributed to the spin contribution to the stress-energy.

The variational approach uses the action principle to relate the variation of the Hilbert action of space-time to that of matter, then invoking Einstein’s equations to identify the matter part to the stress-energy tensor. The result is an expression for the stress-energy which is proportional to the functional derivative of the action of matter, with respect to the inverse metric. The presence of the metric ensures that the stress-energy tensor is manifestly symmetric.

Regardless of the independent nature of their derivations, Goedecke pointed out [8] that in the limit of flat space-time, the Belinfante and variational forms of the stress-energy tensor must agree. The equivalence in the integral spin field case was proven by Rosenfeld [19], and Goedecke provided evidence for equivalence in the half-integral spin field case via a series of examples, but

was not able to provide a general proof. Such a proof for the half-integral case was published shortly afterwards by Lord [17], using the vierbein formalism.

Much effort has been made to “improve” the stress-energy tensor, either by generalizing it beyond the Belinfante/variational forms, or by altering it so that it is compatible with a given theory. An example of the generalization aspect is the work done by Gotay and Marsden, who model the stress-energy in terms of fluxes of the multimomentum map across spacetime hypersurfaces [9]. The Gotay-Marsden stress-energy tensor naturally includes the spin “correction terms” present in the Belinfante formula, but in a more generalized fashion, as well as coinciding with the variational form in the presence of a space-time metric. Work has also been done by Callan, Coleman and Jackiw on making the cut-off dependent symmetric stress-energy tensor compatible with renormalized perturbation theory, by constructing appropriate counter terms in order to make it finite at arbitrarily large cut-off values [3]. The renormalization compatible stress-energy tensor is also compatible with an altered, but phenomenologically consistent, version of general relativity, and simplifies the currents associated with scale and conformal transformations in which the stress-energy appears.

The work we present in this paper can be viewed as analogous to one of Goedecke’s examples, namely the coupled Maxwell-Dirac fields, but in an alternate formalism where the spinor fields are mapped bilinearly to a set of tensor fields.

The background to this approach is as follows. Interrelationships between these “bilinears” can be derived [7] by taking their quadratic products, and expanding out the $\chi\bar{\psi}$ spinor matrices in terms of the Dirac-Clifford algebra basis of 4×4 matrices. For the $\chi \equiv \psi$ case, there are sixteen real, gauge invariant bilinears, equal to the number of basis elements. Comparing this number with the eight real components of the Dirac spinors, and taking the gauge invariance into account, implies that there must be nine independent algebraic equations [14]. Furthermore, the bilinear map is invertible according to the spinor reconstruction theorem by Crawford [4], so given a state described by the set of sixteen bilinear tensor fields, the Dirac spinor field is determined up to a phase. This bilinear reformulation is generalizable to the non-Abelian isospinor representations, at least in the $SU(2)$ doublet case, where an analogous set of Fierz identities involving *sixty four* bilinears can be shown to exist [11].

There are many other identities involving the sixteen bilinears beyond the fundamental set, which can be derived by using the Fierz expansion along with known identities. The extension of this set of sixteen to other classes of bilinears, such as gauge dependent objects where some of the spinors are charge conjugated (i.e., $\chi \equiv \psi^c$), or contain derivatives, is a necessary part of the mathematical framework for the description of the Maxwell-Dirac equations in bilinear form [12]. A serious attempt at generalizing the fundamental set of nine Fierz identities to include a larger set of bilinear “currents” was presented by Takahashi [21], although bilinears containing spinor derivatives were only discussed briefly.

The bilinear representation of electrodynamics is compatible with the Maxwell-Dirac equations, which model the self-consistent behaviour of an electrically charged fermion field interacting with its own electromagnetic field. The derivation of this system relies on the gauge covariant Dirac equation being invertible, with the gauge field, or vector potential A_μ , determined by the state described by the spinors, or equivalently, the bilinears. The inversion in the $U(1)$ (electromagnetic) case was originally considered by Eliezer [6], however he pointed out that the matrix required for the inversion had vanishing determinant, which discounted the possibility for a unique solution. More recently it was shown [2] that if the physical requirement that the vector potential is real is imposed, then the Dirac equation is indeed invertible, provided that a set of consistency conditions are satisfied. Incidentally, a formal inversion of the Dirac equation for the $SU(2)$ Yang-Mills gauge field has also been achieved [11] which extends to $U(2) \cong SU(2) \times U(1)$, but whether or not the inversion can be extended even further to the more physically relevant

$U(1) \times SU(2) \times SU(3)$ case is an open question. The set of equations obtained when the inverted Dirac equation is substituted into the Maxwell equations, along with the appropriate consistency conditions and Fierz identities, constitutes the Maxwell-Dirac equations. We shall not discuss the details of this system further here, other than to state the aim of this paper, which is to derive a bilinear form of the stress-energy tensor for this system, and explore how it reduces in the presence of symmetries. For further information on the bilinear current, or “relativistic hydrodynamical” formulation of the Maxwell-Dirac system, we direct the reader to [20], and [12] where symmetry reductions are also studied.

The first topic we undertake in this paper is the derivation of the bilinear form of the Maxwell-Dirac-Belinfante tensor in Section 2. Following a brief derivation of the Belinfante tensor for a free Dirac particle in the spinor representation, we introduce the core concepts of the bilinear mapping and Fierz expansions. These ideas are then applied to the derivative-dependent bilinear terms that appear naturally in the free Dirac-Belinfante tensor, and an appropriate Fierz identity is derived, allowing us to rewrite the explicitly spinor-dependent form of the tensor exclusively in terms of bilinears. Note that the Fierz transcription to bilinears occurs as the *final stage* of the calculation, once the Belinfante tensor in Dirac spinor form has already been derived. A more detailed version of this derivation is relegated to B. The known tensorial forms of the electromagnetic interaction and Maxwell field stress-energies are then added to the free Dirac contribution, resulting in a manifestly symmetric and gauge independent bilinear form of the Maxwell-Dirac-Belinfante tensor.

Section 3 presents an independent derivation of the Maxwell-Dirac stress-energy tensor, which in this case uses the variational form known from general relativity. Beginning with the Lagrangian density for an electromagnetically interacting Dirac particle, and initially ignoring the Maxwell field contribution since we are mainly interested in the behaviour of the bilinear Dirac contribution, we convert it to its analogous bilinear form, using a contracted form of the Fierz identity obtained in Section 2. This is in stark contrast to the calculational method in Section 2, where the Fierz transcription took place last. A brief review of how the variational stress-energy is obtained is then given. Then, using the general relativistically covariant form of the bilinear Dirac Lagrangian, the variational stress-energy is obtained, and is found to be in agreement with the Maxwell-Dirac-Belinfante tensor.

Finally, in Section 4, the bilinear Maxwell-Dirac stress-energy tensor is subjected to the restrictions imposed by an example symmetry subgroup of the Poincaré group, namely the spherical symmetry group $SO(3)$. This extends the discussion of the symmetry reductions of the Maxwell-Dirac equations from [12], essentially providing the mathematical framework required to calculate the mass-energy and momentum fluxes corresponding to solutions obtained under this symmetry and others, in principle. The discussion of such Maxwell-Dirac solutions and their physical properties, as well as the treatment of other select symmetry groups, are intended to be presented in follow-up works.

2 Maxwell-Dirac stress-energy tensor via Belinfante

2.1 Belinfante tensor for a free Dirac particle

The Belinfante stress-energy tensor is the fully symmetric analogue of the well-known asymmetric “canonical” form, which for free Dirac particles is¹

$$\tilde{T}^{\mu\nu} = -\frac{i}{2} [\bar{\psi}\gamma^\mu(\partial^\nu\psi) - (\partial^\nu\bar{\psi})\gamma^\mu\psi], \quad (1)$$

¹For more details on mathematical conventions, refer to [12] and [13].

that satisfies the conservation condition

$$\partial_\mu \tilde{T}^{\mu\nu} = 0. \quad (2)$$

In fact, $\tilde{T}^{\mu\nu}$ is the Noether symmetry current corresponding to imposing the invariance of the free Dirac Lagrangian

$$\mathcal{L} = \frac{i}{2} [\bar{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi \quad (3)$$

under the translation group. The asymmetry arises from the fact that only translations are considered in the derivation of (1), which neglects rotational contributions to the stress-energy [5]. Since (3) is invariant under Lorentz transformations, we can use the formula for the Noether symmetry current divergence

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta \psi + \delta \bar{\psi} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} \right] = 0, \quad (4)$$

to obtain the Noether current directly. A vanishing, manifestly antisymmetric expression is obtained

$$\frac{1}{4} \partial_\sigma (\bar{\psi} \{ \gamma^\sigma, \sigma^{\mu\nu} \} \psi) + \tilde{T}^{\mu\nu} - \tilde{T}^{\nu\mu} = 0. \quad (5)$$

This can be interpreted as an antisymmetric combination of a symmetric tensor [23], which we call $\Theta^{\mu\nu}$. A tensor form which reduces to the left-hand side of (5) upon antisymmetrization is

$$\Theta^{\mu\nu} = \tilde{T}^{\mu\nu} + \frac{1}{8} \partial_\sigma [\bar{\psi} \{ \gamma^\sigma, \sigma^{\mu\nu} \} \psi - \bar{\psi} \{ \gamma^\mu, \sigma^{\sigma\nu} \} \psi - \bar{\psi} \{ \gamma^\nu, \sigma^{\sigma\mu} \} \psi]. \quad (6)$$

Using (5) to replace the left-most anti-commutator bilinear term, results in the manifestly symmetric combination

$$\Theta^{\mu\nu} = \frac{1}{2} (\tilde{T}^{\mu\nu} + \tilde{T}^{\nu\mu}) - \frac{1}{8} \partial_\sigma [\bar{\psi} \{ \gamma^\mu, \sigma^{\sigma\nu} \} \psi + \bar{\psi} \{ \gamma^\nu, \sigma^{\sigma\mu} \} \psi], \quad (7)$$

where the spin density [19], defined in this case as

$$S^{\mu\sigma\nu} = -\frac{1}{4} [\bar{\psi} \{ \gamma^\mu, \sigma^{\sigma\nu} \} \psi + \bar{\psi} \{ \gamma^\nu, \sigma^{\sigma\mu} \} \psi] = 0, \quad (8)$$

vanishes when taking into account the identity in the Dirac algebra

$$\{ \gamma^\mu, \sigma^{\sigma\nu} \} = 2\epsilon^{\sigma\nu\mu\rho} \gamma_5 \gamma_\rho. \quad (9)$$

We therefore obtain the form of the Belinfante stress-energy tensor for a free Dirac particle

$$\Theta^{\mu\nu} = \frac{1}{2} (\tilde{T}^{\mu\nu} + \tilde{T}^{\nu\mu}) = -\frac{i}{4} [\bar{\psi} \gamma^\mu (\partial^\nu \psi) - (\partial^\nu \bar{\psi}) \gamma^\mu \psi] - \frac{i}{4} [\bar{\psi} \gamma^\nu (\partial^\mu \psi) - (\partial^\mu \bar{\psi}) \gamma^\nu \psi], \quad (10)$$

in agreement with Goedecke [8]. Note that the Belinfante tensor is conserved

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad (11)$$

and is equivalent to the Noether symmetry current of the Poincaré group.

2.2 Dirac bilinears and Fierz identities

The bilinear mapping from the spinor fields $\bar{\chi}$ and ψ to the corresponding set of tensor fields is of the form $\bar{\chi}\Gamma_R\psi$, where $\Gamma_R = \{I, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu, \gamma_5\}$ is the set of $R = 1, \dots, 16$ basis elements of the Dirac Clifford algebra. When $\bar{\chi} = \bar{\psi}$, we obtain the set of manifestly $U(1)$ gauge invariant fields

$$\bar{\psi}\Gamma_R\psi = \{\sigma, j^\mu, s^{\mu\nu}, k^\mu, \omega\}, \quad (12)$$

which are all real except for ω , which is pure imaginary, as is the additional tensor $*s^{\mu\nu} \equiv \bar{\psi}\gamma_5\sigma^{\mu\nu}\psi$. Note that a common alternate convention is to use the real pseudoscalar field $\varpi \equiv \bar{\psi}i\gamma_5\psi$. Alternatively, setting $\bar{\chi} = \bar{\psi}^c$, we obtain for the $\Gamma_R = \gamma^\mu$ element, the gauge dependent bilinear vectors

$$m^\mu + in^\mu = \bar{\psi}^c\gamma^\mu\psi, \quad (13)$$

$$m^\mu = \text{Re}[\bar{\psi}^c\gamma^\mu\psi] = \frac{1}{2}(\bar{\psi}^c\gamma^\mu\psi + \bar{\psi}\gamma^\mu\psi^c), \quad (14)$$

$$n^\mu = \text{Im}[\bar{\psi}^c\gamma^\mu\psi] = \frac{i}{2}(\bar{\psi}\gamma^\mu\psi^c - \bar{\psi}^c\gamma^\mu\psi). \quad (15)$$

The matrix product of two spinors can be expanded in the Γ_R basis, using the Fierz expansion

$$\psi\bar{\chi} = \sum_{R=1}^{16} a_R \Gamma_R = \frac{1}{4}(\bar{\chi}\psi)I + \frac{1}{4}(\bar{\chi}\gamma_\mu\psi)\gamma^\mu + \frac{1}{8}(\bar{\chi}\sigma_{\mu\nu}\psi)\sigma^{\mu\nu} - \frac{1}{4}(\bar{\chi}\gamma_5\gamma_\mu\psi)\gamma_5\gamma^\mu + \frac{1}{4}(\bar{\chi}\gamma_5\psi)\gamma_5, \quad (16)$$

where either $\bar{\chi}$ or ψ may contain partial derivatives. There exists a rich set of interrelationships between bilinear tensor fields, which can be derived by “breaking up” products of bilinears such as $j^\mu k^\nu \equiv \bar{\psi}\gamma^\mu(\psi\bar{\psi})\gamma_5\gamma^\nu\psi$ into sums of different bilinears, by inserting (16). This Fierz expansion technique was used extensively in [12], in the context of the reformulation of the self-coupled Maxwell-Dirac equations in terms of bilinears.

2.3 Belinfante tensor in bilinear form

Our current objective is to rewrite (10) in terms of Fierz bilinears, which means we need to derive a Fierz identity that expresses the spinorial object $[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi]$ in terms of bilinears. Therefore, we are led to search for Fierz expansions in which this term is likely to appear. One example is

$$\begin{aligned} j_\nu[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] &= \frac{i}{3}(\partial_\mu j^\sigma)s_{\nu\sigma} - \frac{i}{3}j^\sigma(\partial_\mu s_{\nu\sigma}) + \frac{1}{3}(\partial_\mu\omega)k_\nu - \frac{1}{3}\omega(\partial_\mu k_\nu) \\ &+ \frac{1}{3}\sigma[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] - \frac{i}{3}s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] \\ &- \frac{i}{3}k^\sigma[\bar{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\sigma_{\nu\sigma}\psi]. \end{aligned} \quad (17)$$

There are at least three other bilinear products whose Fierz expansions produce the desired term, namely $j_\nu[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi]$, $k_\nu(\partial_\mu\sigma)$ and $k_\nu(\partial_\mu\omega)$. Their respective expanded forms, along with a much more detailed derivation of the Fierz identity, is given in appendix B. Using these four identities, we can combine them to give

$$[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] = (\sigma\omega)^{-1} \left(-\frac{i}{2}(\partial_\mu j^\sigma)(\omega s_{\nu\sigma} + \sigma^* s_{\nu\sigma}) - k_\nu[\sigma(\partial_\mu\sigma) + \omega(\partial_\mu\omega)] \right)$$

$$\begin{aligned}
& + \frac{1}{2}(\partial_\mu k_\nu)(\sigma^2 + \omega^2) + j_\nu \{ \omega [\bar{\psi}(\partial_\mu \psi) - (\partial_\mu \bar{\psi})\psi] + \sigma [\bar{\psi} \gamma_5 (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma_5 \psi] \} \\
& + \frac{i}{2}(\sigma s_{\nu\sigma} + \omega^* s_{\nu\sigma}) [\bar{\psi} \gamma_5 \gamma^\sigma (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma_5 \gamma^\sigma \psi] \Big), \tag{18}
\end{aligned}$$

which obviously still requires some more work. Using the Fierz identities derived in [12]

$$[\bar{\psi}(\partial_\mu \psi) - (\partial_\mu \bar{\psi})\psi] = -(\sigma^2 - \omega^2)^{-1} [j^\nu (\partial_\mu k_\nu) \omega + i m^\nu (\partial_\mu n_\nu) \sigma], \tag{19}$$

$$[\bar{\psi} \gamma_5 (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma_5 \psi] = -(\sigma^2 - \omega^2)^{-1} [j^\nu (\partial_\mu k_\nu) \sigma + i m^\nu (\partial_\mu n_\nu) \omega], \tag{20}$$

we can replace the spinor terms on the second line of (18) with bilinears, but this still leaves the spinor terms on the third line. After a straightforward, but tedious, set of Fierz manipulations, we obtain the desired identity

$$\begin{aligned}
\frac{i}{2}(\sigma s_{\nu\sigma} + \omega^* s_{\nu\sigma}) [\bar{\psi} \gamma_5 \gamma^\sigma (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma_5 \gamma^\sigma \psi] &= -\frac{1}{2} \sigma \omega [\bar{\psi} \gamma_\nu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma_\nu \psi] \\
&+ \frac{1}{4} j_\nu (\sigma^2 - \omega^2)^{-1} [j^\sigma (\partial_\mu k_\sigma) (\sigma^2 + \omega^2) + 2i m^\sigma (\partial_\mu n_\sigma) \sigma \omega] - \frac{3}{8} (\sigma^2 + \omega^2) (\partial_\mu k_\nu) \\
&+ \frac{3}{8} k_\nu [\sigma (\partial_\mu \sigma) + \omega (\partial_\mu \omega)] - \frac{i}{8} (\partial_\mu j^\sigma) (\sigma^* s_{\nu\sigma} + \omega s_{\nu\sigma}) + \frac{i}{8} j^\sigma [\sigma (\partial_\mu^* s_{\nu\sigma}) + \omega (\partial_\mu s_{\nu\sigma})]. \tag{21}
\end{aligned}$$

Substituting into (18) and rearranging yields

$$\begin{aligned}
[\bar{\psi} \gamma_\nu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma_\nu \psi] &= (\sigma \omega)^{-1} \left\{ -\frac{1}{2} j_\nu (\sigma^2 - \omega^2)^{-1} [j^\sigma (\partial_\mu k_\sigma) (\sigma^2 + \omega^2) + 2i m^\sigma (\partial_\mu n_\sigma) \sigma \omega] \right. \\
&+ \frac{1}{12} (\sigma^2 + \omega^2) (\partial_\mu k_\nu) - \frac{5}{12} k_\nu [\sigma (\partial_\mu \sigma) + \omega (\partial_\mu \omega)] + \frac{i}{12} j^\sigma [\sigma (\partial_\mu^* s_{\nu\sigma}) + \omega (\partial_\mu s_{\nu\sigma})] \\
&\left. - \frac{5i}{12} (\partial_\mu j^\sigma) (\sigma^* s_{\nu\sigma} + \omega s_{\nu\sigma}) \right\}. \tag{22}
\end{aligned}$$

This is entirely in terms of bilinears, but we would like to go further and eliminate the rank-2 terms, $s_{\mu\nu}$ and $s_{\mu\nu}^*$. Using the known Fierz identities for the replacement of these terms [14]

$$s^{\mu\nu} = (\sigma^2 - \omega^2)^{-1} (\sigma \epsilon^{\mu\nu\rho\sigma} - \omega \delta^{\mu\nu\rho\sigma}) j_\rho k_\sigma, \tag{23}$$

$$s^{\mu\nu*} = (\sigma^2 - \omega^2)^{-1} (\omega \epsilon^{\mu\nu\rho\sigma} - \sigma \delta^{\mu\nu\rho\sigma}) j_\rho k_\sigma, \tag{24}$$

where we define the partially antisymmetric object

$$\delta^{\mu\nu\rho\sigma} \equiv i(\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}), \tag{25}$$

the rank-2 dependent terms in (22) become

$$\begin{aligned}
\frac{i}{12} j^\sigma [\sigma (\partial_\mu^* s_{\nu\sigma}) + \omega (\partial_\mu s_{\nu\sigma})] &= \frac{1}{12} (\sigma^2 - \omega^2)^{-1} \{ 2\sigma \omega k_\nu [\omega (\partial_\mu \sigma) - \sigma (\partial_\mu \omega)] \\
&+ 2i \sigma \omega \epsilon_{\nu\sigma\rho\epsilon} j^\sigma (\partial_\mu j^\rho) k^\epsilon + j_\nu j^\sigma (\partial_\mu k_\sigma) (\sigma^2 + \omega^2) \} - \frac{1}{12} (\partial_\mu k_\nu) (\sigma^2 + \omega^2), \tag{26}
\end{aligned}$$

and

$$-\frac{5i}{12} (\partial_\mu j^\sigma) (\sigma^* s_{\nu\sigma} + \omega s_{\nu\sigma}) = (\sigma^2 - \omega^2)^{-1} \left\{ \frac{5}{12} j_\nu j^\sigma (\partial_\mu k_\sigma) (\sigma^2 + \omega^2) - \frac{5i}{6} \sigma \omega \epsilon_{\nu\sigma\rho\epsilon} (\partial_\mu j^\sigma) j^\rho k^\epsilon \right.$$

$$-\frac{5}{12}k_\nu(\sigma^2 + \omega^2)[\omega(\partial_\mu\omega) - \sigma(\partial_\mu\sigma)]\}, \quad (27)$$

giving us the final form of our identity

$$[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] = (\sigma^2 - \omega^2)^{-1}\{k_\nu[\omega(\partial_\mu\sigma) - \sigma(\partial_\mu\omega)] - i\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma)j^\rho k^\epsilon - j_\nu m^\sigma(\partial_\mu n_\sigma)\}. \quad (28)$$

Substituting into (10) and relabelling some indices for convenience later on, we obtain the bilinear form of the Belinfante stress-energy tensor for a free Dirac particle

$$\Theta_{\mu\nu,D} = \frac{1}{4}(\sigma^2 - \omega^2)^{-1}\{-i[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] - j_\sigma k_\kappa[\epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho) + \epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)] - j_\mu m^\sigma(\partial_\nu n_\sigma) - j_\nu m^\sigma(\partial_\mu n_\sigma)\}. \quad (29)$$

The Fierz identity (28) is of central importance to our current work on the stress-energy tensor, since it permits an explicit mapping from the Dirac spinor fields to the physically equivalent set of bilinears, in the form in which the spinors appear in the Belinfante tensor (10). Of note is that none of the elements of the bilinear set (12) which we have chosen to work with contain any *internal* derivative operators, so all gradient terms appear *explicitly* in (29). Additionally, (28), along with (19) and (20), are examples of a large family of related Fierz identities, and only represent the tip of the iceberg of such relations [21].

2.4 Maxwell-Dirac Belinfante tensor

The full Maxwell-Dirac stress-energy tensor is

$$\Theta_{\mu\nu,MD} = \Theta_{\mu\nu,D} + \Theta_{\mu\nu,int} + \Theta_{\mu\nu,em}, \quad (30)$$

where $\Theta_{\mu\nu,int}$ and $\Theta_{\mu\nu,em}$ are the interaction and Maxwell field contributions respectively. The Maxwell contribution has the well known form

$$\Theta_{\mu\nu,em} = \frac{1}{4}\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - F_{\mu\sigma}F_\nu{}^\sigma, \quad (31)$$

consistent with our metric signature $(+ - - -)$, and the interaction term is

$$\Theta_{\mu\nu,int} = \frac{q}{2}(j_\mu A_\nu + j_\nu A_\mu), \quad (32)$$

where the electromagnetic vector potential A_μ can be replaced by the gauge independent analogue B_μ using the definition from [12]

$$A_\mu = B_\mu + \frac{1}{2q}(\sigma^2 - \omega^2)^{-1}m^\sigma(\partial_\mu n_\sigma). \quad (33)$$

The gauge dependent bilinear terms in (32) cancel out the corresponding terms in (29) exactly, so the full Maxwell-Dirac Belinfante stress-energy tensor is

$$\begin{aligned} \Theta_{\mu\nu,MD} = & \frac{1}{4}(\sigma^2 - \omega^2)^{-1}\{-i[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] - j_\sigma k_\kappa[\epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho) + \epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)]\} \\ & + \frac{q}{2}(j_\mu B_\nu + j_\nu B_\mu) + \frac{1}{4}\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - F_{\mu\sigma}F_\nu{}^\sigma, \end{aligned} \quad (34)$$

which is manifestly symmetric and gauge independent.

3 Maxwell-Dirac stress-energy tensor via general relativity

3.1 Bilinear form of Dirac Lagrangian

We will now derive the bilinear form of the Maxwell-Dirac stress-energy tensor again, this time using a completely different method. The approach outlined in section 2 involved the use of Fierz identities to convert the spinorial form of the Belinfante stress-energy for a free Dirac particle (10) into bilinear form (29), to which the known tensor forms of the interaction and Maxwell contributions were added, yielding (34).

This time around, we convert the Lagrangian for an interacting Dirac particle

$$\mathcal{L} = \frac{i}{2} [\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi - q\bar{\psi}\gamma^\mu\psi A_\mu, \quad (35)$$

into bilinear form, then use the definition of the stress-energy tensor from general relativity to obtain our result, which should in principle agree with (34). Note that this method from general relativity was used directly on the spinorial form of the Lagrangian (3) by Goedecke [8], who then demonstrated its equivalence with the Belinfante stress-energy tensor (10). We are pursuing a similar equivalence demonstration, with our focus being on the bilinear formalism. However, in contrast with the calculation of the previous section where the Fierz bilinear transcription was performed at the end, once the spinorial Belinfante tensor (10) had been obtained, here we transcribe *at the start*, and perform the entire variational calculation with the bilinear field variables. The Fierz identity required to rewrite (35) is obtained by simply substituting the contracted form of (28),

$$[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] = (\sigma^2 - \omega^2)^{-1} \{k^\mu[\omega(\partial_\mu\sigma) - \sigma(\partial_\mu\omega)] - i\epsilon^{\mu\sigma\rho\epsilon}(\partial_\mu j_\sigma)j_\rho k_\epsilon - ij^\mu m^\sigma(\partial_\mu n_\sigma)\}. \quad (36)$$

Applying the definitions $\sigma \equiv \bar{\psi}\psi$, $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$, and using the definition of B_μ (33), we obtain

$$\mathcal{L} = \frac{1}{2}(\sigma^2 - \omega^2)^{-1} \{ik^\rho[\omega(\partial_\rho\sigma) - \sigma(\partial_\rho\omega)] + \epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau\} - m\sigma - qj^\rho B_\rho. \quad (37)$$

3.2 Variational form of the stress-energy tensor

The total action for the gravitational field in the presence of matter is [22]

$$S = \frac{S_H}{16\pi G} + S_M, \quad (38)$$

where S_M is the action for matter fields (mass-energy). S_H is the Hilbert action, defined as

$$S_H = \int d^4x \sqrt{-g} R, \quad (39)$$

where g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, and d^4x is the invariant volume element. The variation of the action with respect to an arbitrary tensor field $\Phi^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$ takes the general form

$$\delta S = \int d^4x \frac{\delta S}{\delta \Phi} \delta \Phi, \quad (40)$$

with contraction over the indices implied. The term $\delta S/\delta\Phi$ is called the functional derivative of S with respect to the tensor field Φ . Of main interest in variational theory are tensors Φ_0 which extremize the action, so that $\delta S = 0$, and hence

$$\left. \frac{\delta S}{\delta\Phi} \right|_{\Phi_0} = 0. \quad (41)$$

Extremizing the variation of the Hilbert action (39) with respect to the inverse metric leads to Einstein's equations in vacuum

$$\frac{1}{\sqrt{-g}} \frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \quad (42)$$

Likewise, extremizing the gravitational action in the presence of matter (38), so that

$$\delta S = \frac{\delta S_H}{16\pi G} + \delta S_M = 0, \quad (43)$$

and equating the corresponding functional derivatives, yields

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \quad (44)$$

Comparing with the well-known form of Einstein's equations in the presence of matter

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (45)$$

we can see that the energy momentum tensor is of the general form

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \quad (46)$$

3.3 Variational Maxwell-Dirac stress-energy tensor

The variation of the electromagnetically interacting Dirac matter action, ignoring the contribution of the Maxwell field itself for now, is given by

$$\delta S = \int d^4x \left(\delta\sqrt{-g} \mathcal{L} + \sqrt{-g} \delta\mathcal{L} \right), \quad (47)$$

where the Lagrangian is given by (37), but with appropriate modifications to make it covariant in curved space. Since the invariant volume element d^4x and $\sqrt{-g}$ are scalar densities of weight -1 and $+1$ respectively, we must arrange for the Lagrangian to be manifestly a scalar. Notice that (37) contains a term dependent on the Levi-Civita symbol with upstairs indices, which is of weight -1 . This implies that we should make the replacement

$$\epsilon^{\rho\sigma\kappa\tau} \rightarrow \frac{1}{\sqrt{-g}} \epsilon^{\rho\sigma\kappa\tau}. \quad (48)$$

In order to deal with the bilinear four-vectors we must introduce the vierbein fields [24], which locally relate the curved metric to the flat one

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad (49)$$

where Greek and Latin indices label curved and flat spacetime components respectively. The gamma matrices are modified such that

$$\gamma_\mu = e_\mu^a \gamma_a; \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad (50)$$

so the bilinears are now

$$j_\mu = e_\mu^a j_a; \quad j_a = \bar{\psi} \gamma_a \psi, \quad (51)$$

$$k_\mu = e_\mu^a k_a; \quad k_a = \bar{\psi} \gamma_5 \gamma_a \psi. \quad (52)$$

The variation of the square root of the negative metric determinant is [22]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \quad (53)$$

Noting that $h = \sqrt{-g}$, where h is the vierbein determinant, we can use the variation of (49) to alternatively write this as

$$\delta h = -h e_\mu^a (\delta e^\mu_a), \quad (54)$$

implying the reciprocal variation

$$\delta(h^{-1}) = h^{-1} e_\mu^a (\delta e^\mu_a). \quad (55)$$

In curved space, the Levi-Civita term in (37) becomes

$$h^{-1} \epsilon^{\rho\sigma\kappa\tau} (\partial_\rho e_\sigma^a) j_a j_\kappa k_\tau + h^{-1} \epsilon^{\rho\sigma\kappa\tau} e_\sigma^a (\partial_\rho j_a) j_\kappa k_\tau. \quad (56)$$

Introducing the covariant derivative causes the first term to vanish, due to the tetrad postulate [24]

$$\nabla_\mu e_\nu^a = 0. \quad (57)$$

Expanding out all of the vierbein fields in the second term, we find that

$$\epsilon^{\rho\sigma\kappa\tau} e_\sigma^a (\partial_\rho j_a) j_\kappa k_\tau = \epsilon^{abcd} (\partial_a j_b) j_c k_d, \quad (58)$$

which implies that for any curved coordinate components, this term is always equal to the flat spacetime version, so it is automatically covariant. We find that the covariant bilinear electromagnetically interacting Dirac matter Lagrangian has the form

$$\mathcal{L} = \frac{1}{2}(\sigma^2 - \omega^2)^{-1} [\text{ig}^{\rho\sigma} e_\sigma^a k_a (\omega \partial_\rho \sigma - \sigma \partial_\rho \omega) + h^{-1} \epsilon^{\rho\sigma\kappa\tau} e_\sigma^a e_\kappa^b e_\tau^c (\partial_\rho j_a) j_b k_c] - m\sigma - qg^{\rho\sigma} e_\sigma^a j_a B_\rho. \quad (59)$$

The variation with respect to deformation of the vierbein field is

$$\begin{aligned} \delta\mathcal{L} = & \frac{1}{2}(\sigma^2 - \omega^2)^{-1} \{ \text{i} \delta(g^{\rho\sigma} e_\sigma^a) k_a (\omega \partial_\rho \sigma - \sigma \partial_\rho \omega) + \epsilon^{\rho\sigma\kappa\tau} \delta(h^{-1} e_\sigma^a e_\kappa^b e_\tau^c) (\partial_\rho j_a) j_b k_c \} \\ & - q \delta(g^{\rho\sigma} e_\sigma^a) j_a B_\rho. \end{aligned} \quad (60)$$

From the variation of (49), we find that

$$\delta(g^{\rho\sigma} e_\sigma^a) = 2(\delta e^{\rho a}) + (\delta e^\sigma_b) e^{\rho b} e_\sigma^a. \quad (61)$$

Using the fundamental vierbein property

$$e_\mu^a e^\mu_b = \delta_b^a, \quad (62)$$

$$e_\mu^a (\delta e^\mu_b) = -(\delta e_\mu^a) e^\mu_b, \quad (63)$$

we find that the first and last terms in (60) are

$$i\delta(g^{\rho\sigma} e_\sigma^a) k_a (\omega \partial_\rho \sigma - \sigma \partial_\rho \omega) = -i k_\mu (\omega \partial^a \sigma - \sigma \partial^a \omega) (\delta e^\mu_a), \quad (64)$$

$$-q\delta(g^{\rho\sigma} e_\sigma^a) j_a B_\rho = q j_\mu B^a (\delta e^\mu_a). \quad (65)$$

Following a similar process, we find that the second variational term is

$$\begin{aligned} & \epsilon^{\rho\sigma\kappa\tau} \delta(h^{-1} e_\sigma^a e_\kappa^b e_\tau^c) (\partial_\rho j_a) j_b k_c \\ &= h^{-1} [-e_\mu^a \epsilon^{\rho\sigma\kappa\tau} e_\sigma^d (\partial_\rho j_d) j_\kappa k_\tau + \epsilon^\rho{}_\mu{}^\sigma{}_\kappa (\partial_\rho j^a) j_\sigma k_\kappa + \epsilon^{\rho\sigma}{}_\mu{}^\kappa e_\sigma^b (\partial_\rho j_b) j^a k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\mu e_\sigma^b (\partial_\rho j_b) j_\kappa k^a] (\delta e^\mu_a). \end{aligned} \quad (66)$$

Gathering the deformed terms together, we can write the variation of the Lagrangian as

$$\begin{aligned} \delta\mathcal{L} = & \left(\frac{1}{2} (\sigma^2 - \omega^2)^{-1} \{ -i k_\mu (\omega \partial^a \sigma - \sigma \partial^a \omega) + h^{-1} [-e_\mu^a \epsilon^{\rho\sigma\kappa\tau} e_\sigma^d (\partial_\rho j_d) j_\kappa k_\tau \right. \\ & \left. + \epsilon^\rho{}_\mu{}^\sigma{}_\kappa (\partial_\rho j^a) j_\sigma k_\kappa + \epsilon^{\rho\sigma}{}_\mu{}^\kappa e_\sigma^b (\partial_\rho j_b) j^a k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\mu e_\sigma^b (\partial_\rho j_b) j_\kappa k^a \} + q j_\mu B^a \right) (\delta e^\mu_a), \end{aligned} \quad (67)$$

with the associated action variation being

$$\begin{aligned} \delta S_D = & \int d^4x \sqrt{-g} \left(-e_\mu^a \left\{ \frac{1}{2} (\sigma^2 - \omega^2)^{-1} [i k^\rho (\omega \partial_\rho \sigma - \sigma \partial_\rho \omega) + \epsilon^{\rho\sigma\kappa\tau} (\partial_\rho j_\sigma) j_\kappa k_\tau] - m\sigma - q j^\rho B_\rho \right\} \right. \\ & + \frac{1}{2} (\sigma^2 - \omega^2)^{-1} \{ -i k_\mu (\omega \partial^a \sigma - \sigma \partial^a \omega) + h^{-1} [-e_\mu^a \epsilon^{\rho\sigma\kappa\tau} e_\sigma^d (\partial_\rho j_d) j_\kappa k_\tau + \epsilon^\rho{}_\mu{}^\sigma{}_\kappa (\partial_\rho j^a) j_\sigma k_\kappa \\ & \left. + \epsilon^{\rho\sigma}{}_\mu{}^\kappa e_\sigma^b (\partial_\rho j_b) j^a k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\mu e_\sigma^b (\partial_\rho j_b) j_\kappa k^a \} + q j_\mu B^a \right) (\delta e^\mu_a). \end{aligned} \quad (68)$$

From the general form of the action variation (40), a relationship between (68) and the stress-energy tensor can be obtained [24]

$$\delta S_D = \int d^4x \sqrt{-g} u_\lambda^a \delta e^\lambda_a = \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}, \quad (69)$$

which implies that

$$u_\mu^a = \frac{1}{2} (T_{\mu\lambda} e^{\lambda a} + T_{\lambda\mu} e^{\lambda a}). \quad (70)$$

Recognizing $T_{\mu\nu}$ as symmetric gives

$$T_{\mu\nu} = \frac{1}{2} (e_{\mu a} u_\nu^a + e_{\nu a} u_\mu^a). \quad (71)$$

Identifying the contents of the external parentheses in (68) with u_μ^a , we obtain for the stress energy tensor

$$T_{\mu\nu} = -\eta_{\mu\nu} \left\{ \frac{1}{2} (\sigma^2 - \omega^2)^{-1} [i k^\rho (\omega \partial_\rho \sigma - \sigma \partial_\rho \omega) + \epsilon^{\rho\sigma\kappa\tau} (\partial_\rho j_\sigma) j_\kappa k_\tau] - m\sigma - q j^\rho B_\rho \right\}$$

$$\begin{aligned}
& + \frac{1}{4}(\sigma^2 - \omega^2)^{-1} \{ -i[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] - 2\eta_{\mu\nu}\epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau \\
& + \epsilon^\rho{}_\mu{}^{\sigma\kappa}(\partial_\rho j_\nu)j_\sigma k_\kappa + \epsilon^{\rho\sigma}{}_\mu{}^\kappa(\partial_\rho j_\sigma)j_\nu k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\mu(\partial_\rho j_\sigma)j_\kappa k_\nu + \epsilon^\rho{}_\nu{}^{\sigma\kappa}(\partial_\rho j_\mu)j_\sigma k_\kappa \\
& + \epsilon^{\rho\sigma}{}_\nu{}^\kappa(\partial_\rho j_\sigma)j_\mu k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\nu(\partial_\rho j_\sigma)j_\kappa k_\mu \} + \frac{q}{2}(j_\mu B_\nu + j_\nu B_\mu), \tag{72}
\end{aligned}$$

where we have evaluated at flat spacetime. This is manifestly symmetric, but it requires some additional manipulation before it more closely resembles the Belinfante form (34). Consider the $U(1)$ gauge covariant Dirac equation and its Dirac conjugate

$$i\gamma^\sigma(\partial_\sigma\psi) - q\gamma^\sigma A_\sigma\psi - m\psi = 0, \tag{73}$$

$$i(\partial_\sigma\bar{\psi})\gamma^\sigma + q\bar{\psi}\gamma^\sigma A_\sigma + m\bar{\psi} = 0. \tag{74}$$

Left and right multiplying these equations by $\bar{\psi}$ and ψ respectively, then subtracting the second from the first and rearranging, gives

$$m\sigma = \frac{i}{2} [\bar{\psi}\gamma^\sigma(\partial_\sigma\psi) - (\partial_\sigma\bar{\psi})\gamma^\sigma\psi] - qj^\sigma A_\sigma \tag{75}$$

Applying the Fierz identity (36) and the B_μ definition (33), this becomes

$$-m\sigma = -\frac{1}{2}(\sigma^2 - \omega^2)^{-1} [ik^\rho(\omega\partial_\rho\sigma - \sigma\partial_\rho\omega) + \epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau] - qj^\rho B_\rho, \tag{76}$$

causing the $\eta_{\mu\nu}$ dependent term in (72) to vanish. Now consider the combinatorial identity²

$$\begin{aligned}
& -\epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)j_\sigma k_\kappa - \epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho)j_\sigma k_\kappa \\
& = -2\eta_{\mu\nu}\epsilon^{\rho\sigma\kappa\tau}(\partial_\rho j_\sigma)j_\kappa k_\tau + \epsilon^\rho{}_\mu{}^{\sigma\kappa}(\partial_\rho j_\nu)j_\sigma k_\kappa + \epsilon^{\rho\sigma}{}_\mu{}^\kappa(\partial_\rho j_\sigma)j_\nu k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\mu(\partial_\rho j_\sigma)j_\kappa k_\nu \\
& \quad + \epsilon^\rho{}_\nu{}^{\sigma\kappa}(\partial_\rho j_\mu)j_\sigma k_\kappa + \epsilon^{\rho\sigma}{}_\nu{}^\kappa(\partial_\rho j_\sigma)j_\mu k_\kappa + \epsilon^{\rho\sigma\kappa}{}_\nu(\partial_\rho j_\sigma)j_\kappa k_\mu, \tag{77}
\end{aligned}$$

which can be used to obtain the final form of the variational stress-energy tensor for Dirac matter

$$\begin{aligned}
T_{\mu\nu,D} &= \frac{1}{4}(\sigma^2 - \omega^2)^{-1} \{ -i[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] - j_\sigma k_\kappa [\epsilon_\nu{}^{\rho\sigma\kappa}(\partial_\mu j_\rho) + \epsilon_\mu{}^{\rho\sigma\kappa}(\partial_\nu j_\rho)] \} \\
& \quad + \frac{q}{2}(j_\mu B_\nu + j_\nu B_\mu). \tag{78}
\end{aligned}$$

Comparing with the Belinfante tensor (34), we find that they agree

$$T_{\mu\nu,MD} = \Theta_{\mu\nu,MD}, \tag{79}$$

when the gauge field stress-energy (31) is included on the left-hand side.

4 Symmetry reduction of the Maxwell-Dirac stress-energy tensor

The Maxwell-Dirac equations in the bilinear formalism are in general, a very complicated set of self-coupled partial differential equations. An application of the present construction of the

²This follows from the 5 term cyclic identity $V^\alpha\epsilon^{\rho\sigma\kappa\tau} + V^\tau\epsilon^{\alpha\rho\sigma\kappa} + \dots = 0$ which holds for the Levi-Civita tensor multiplied by any contravariant vector quantity. With the role of V^α played by the Kronecker δ_β^α (for fixed β), this yields (77) after contracting with $\eta_{\mu\alpha}\delta_\nu^\beta(\partial_\rho j_\sigma)j_\kappa k_\tau$, and rearranging indices appropriately.

physical stress-energy tensor of the system in terms of bilinears, is therefore to provide a representation of the conserved rest mass of possible solutions (via the spatial integral of T_{00} for example).

For any meaningful solutions to be derived, it is natural to consider reduced forms under the imposition of special symmetries. The reduction of the bilinear form of the Maxwell-Dirac system under several examples of subgroups of the Poincaré group was discussed in [12]. We therefore choose one of the most important of these subgroups to work with here, namely $SO(3)$, which corresponds to spherical symmetry. In particular, we shall demonstrate how the bilinear form of the Maxwell-Dirac stress-energy tensor reduces, given the restrictions imposed by this subgroup. The treatment of other symmetry reductions, such as cylindrical symmetry and the $\tilde{P}_{13,10}$ subgroup from [18], shall be left for future work.

Under spherical symmetry, scalar fields (σ, ω , etc.) have the generic form

$$\phi = f(t, r) \quad (80)$$

and vector fields (j^μ, k^μ , etc.) have the form

$$\Phi^\mu = \begin{pmatrix} f(t, r) \\ xg(t, r) \\ yg(t, r) \\ zg(t, r) \end{pmatrix}, \quad (81)$$

where the invariant

$$r = \sqrt{x^2 + y^2 + z^2} \quad (82)$$

is simply the spatial radius. We showed in [12] that the spherically symmetric forms of our bilinear vector and axial vector fields are

$$j^\mu = \begin{pmatrix} j_a \\ xj_b \\ yj_b \\ zj_b \end{pmatrix}, \quad k^\mu = \begin{pmatrix} rj_b \\ (x/r)j_a \\ (y/r)j_a \\ (z/r)j_a \end{pmatrix}, \quad B^\mu = \begin{pmatrix} B_a \\ xB_b \\ yB_b \\ zB_b \end{pmatrix}, \quad (83)$$

where the vector potential functions are

$$B_a = \left[\pm \frac{i}{2} (\sigma_r \omega - \sigma \omega_r) - m \sigma j_a \right] [q(\sigma^2 - \omega^2)]^{-1}, \quad (84)$$

$$B_b = \left[\mp \frac{i}{2r} (\sigma_t \omega - \sigma \omega_t) - m \sigma j_b \right] [q(\sigma^2 - \omega^2)]^{-1}. \quad (85)$$

Here, we are using a condensed derivative notation $\partial_t \sigma \equiv \sigma_t$, and so on. Note that the effect of the symmetry reduction has in this case, reduced the components of the four-vectors to the set of coefficient functions j_a, j_b, σ and ω , which are themselves further constrained by higher-order nonlinear PDEs in the Maxwell-Dirac system. The coefficient functions k_a and k_b have been eliminated through the use of the Fierz identities

$$j_\mu j^\mu = -k_\mu k^\mu = \sigma^2 - \omega^2, \quad (86)$$

$$j_\mu k^\mu = 0. \quad (87)$$

It is straightforward to show that the Levi-Civita terms in the stress-energy vanish in this symmetry case

$$j_\sigma k_\kappa [\epsilon_\nu^{\rho\sigma\kappa} (\partial_\mu j_\rho) + \epsilon_\mu^{\rho\sigma\kappa} (\partial_\nu j_\rho)] = 0. \quad (88)$$

The form of the stress-energy tensor we are dealing with is therefore

$$T_{\mu\nu, \text{MD}} = -\frac{i}{4}(\sigma^2 - \omega^2)^{-1}[k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) + k_\nu(\omega\partial_\mu\sigma - \sigma\partial_\mu\omega)] + \frac{q}{2}(j_\mu B_\nu + j_\nu B_\mu) + \frac{1}{4}\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} - F_{\mu\sigma}F_\nu{}^\sigma. \quad (89)$$

For the $SO(3)$ symmetry, the components of the electromagnetic field strength tensor are

$$F_{0i} = x_i F_a(t, r), \quad (90)$$

$$F_{ij} = \epsilon_{ijk} x^k F_b(t, r), \quad (91)$$

where the Maxwell coefficient functions are

$$F_a(t, r) = (1/qr)(\sigma^2 - \omega^2)^{-2}\{-2m[\sigma j_a(\sigma\sigma_r - \omega\omega_r) + r\sigma j_b(\sigma\sigma_t - \omega\omega_t)] \pm i[\sigma\omega(\sigma_r^2 - \sigma_t^2 + \omega_r^2 - \omega_t^2) + (\sigma^2 + \omega^2)(\sigma_t\omega_t - \sigma_r\omega_r)]\} + (1/qr)(\sigma^2 - \omega^2)^{-1}[m(\sigma_r j_a + \sigma j_{a,r} + r\sigma_t j_b + r\sigma j_{b,t}) \pm (i/2)(\sigma_{tt}\omega - \sigma\omega_{tt} - \sigma_{rr}\omega + \sigma\omega_{rr})], \quad (92)$$

representing the electric field form, and

$$F_b(r) = \pm \frac{1}{2qr^3}, \quad (93)$$

representing the magnetic field form, which happens to be that of a monopole. Treating the $\mu = \nu = 0$, $\mu = 0$, $\nu = i$ and $\mu = i$, $\nu = j$ cases separately, we find that the respective components of (89) are

$$T_{00, \text{MD}} = T_a + \mathcal{F}, \quad (94)$$

$$T_{0i, \text{MD}} = \frac{x_i}{r} T_b, \quad (95)$$

$$T_{ij, \text{MD}} = \frac{x_i x_j}{r^2} T_c + \delta_{ij} \mathcal{F}, \quad (96)$$

where the energy density of the Maxwell field is

$$\mathcal{F} = \frac{r^2(F_a^2 + F_b^2)}{2}, \quad (97)$$

and the other functions are defined as

$$T_a(t, r) = (\sigma^2 - \omega^2)^{-1} \left\{ \pm \frac{i}{2} [j_a(\sigma_r\omega - \sigma\omega_r) - rj_b(\sigma_t\omega - \sigma\omega_t)] - m\sigma j_a^2 \right\}, \quad (98)$$

$$T_b(t, r) = (\sigma^2 - \omega^2)^{-1} \left\{ \pm \frac{i}{2} [j_a(\sigma_t\omega - \sigma\omega_t) - rj_b(\sigma_r\omega - \sigma\omega_r)] - m\sigma j_a r j_b \right\}, \quad (99)$$

$$T_c(t, r) = (\sigma^2 - \omega^2)^{-1} \left\{ \pm \frac{i}{2} [j_a(\sigma_r\omega - \sigma\omega_r) - rj_b(\sigma_t\omega - \sigma\omega_t)] - m\sigma r^2 j_b^2 \right\} - 2\mathcal{F}. \quad (100)$$

5 Conclusions

The central aim of our current work was to obtain a form of the stress-energy tensor for the self-coupled Maxwell-Dirac system, in terms of bilinears. The motivation for working in the bilinear formalism in general, is given by the fact that spinor fields do not correspond directly to

observables, due to their dependence on an arbitrary unobservable phase factor. On the other hand, the system of Dirac bilinears corresponds to “density fields”, or “probability distributions”, which are observable according to a fundamental postulate of quantum mechanics. However, the attractiveness of the new physical insights that could be gained by studying the fully non-linear quantum electrodynamics, in terms of the observable tensor fields given by bilinears [20], is offset somewhat by the sheer complexity of the self-coupled system. Reducing this system under a symmetry subgroup of the Poincaré group can simplify it to a point where it is mathematically tractable, and solutions can potentially be obtained. Such solutions would have associated mass-energy and momentum fluxes, which can be calculated directly using a bilinear-dependent form of the symmetry reduced stress-energy tensor.

In sections 2 and 3, we demonstrated that bilinear forms for the stress-energy tensor ($\Theta_{\mu\nu}$ and $T_{\mu\nu}$ respectively) can indeed be calculated, by applying Fierz identities to the spinor terms appearing in the Belinfante and variational general relativistic calculational schemes. In the Belinfante case, the Fierz mapping was applied to the spinorial Belinfante tensor, and in the variational case, it was applied to the spinorial Lagrangian prior to the vierbein deformation. Despite the fact that these two methods are quite independent, with very different starting points from the point of view of the Fierz bilinear transcription, they are in agreement in accordance with Goedecke’s conjecture [8] and Lord’s subsequent equivalence proof [17]. This mathematical consistency adds weight to the validity of the bilinear representation of the Maxwell-Dirac system, and our corresponding bilinear stress-energy result.

However, there is a point of view from which this automatic agreement is somewhat surprising. When taking into consideration the details of the functional Jacobian corresponding to the spinor to bilinear mapping, one would expect there to be extra constraint terms entering into the bilinearized Lagrangian, with the lack of such terms in (37) leading to a disagreement with the Belinfante tensor in the *bilinear* representation. A transcription of spinor electrodynamics into gauge invariant quantities in this spirit, has been given in the functional formalism by Rudolph and Kijowski [15], [16]. In their bosonic transcription, Green’s functions are given as functional integrals in whose integrands there are always additional accompanying field-dependent factors, and so an effective bosonic, local, purely Lagrangian formulation is not obtained. The details of the agreement between $\Theta_{\mu\nu}$ and $T_{\mu\nu}$ for the bilinear case, although highly encouraging, remains a matter deserving of further study.

Putting these technical concerns aside, we then turned to an example to demonstrate how the bilinear stress-energy tensor is reduced under spherical symmetry, using the generic $SO(3)$ invariant forms for scalar and four-vector fields discussed in [12]. We found that the stress-energy components could be described in terms of three functions (98)-(100) corresponding to the interacting Dirac matter contribution, as well as a single function (97), corresponding to the energy density of the Maxwell field.

The bilinear form of the Maxwell-Dirac stress-energy tensor (34) may be easily applied to *solutions* of the symmetry reduced Maxwell-Dirac equations, such as considered in [12], by performing the corresponding symmetry reduction on $T_{\mu\nu}$, as done in section 4 for the $SO(3)$ group, and directly substituting the solution fields. We intend to investigate both numerical and closed-form solutions explicitly, for the static spherically symmetric and the algebraic splitting groups $\tilde{P}_{13,10}$ [12], [18], in follow-up works.

Acknowledgements

This study was financially supported by the Australian Postgraduate Awards (APA) program.

A Dirac identities

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (101)$$

$$[\gamma^\mu, \gamma^\nu] = -2i\sigma^{\mu\nu} \quad (102)$$

$$\gamma^5 = \gamma_5 = -(i/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \quad (103)$$

$$\gamma_5^2 = I \quad (104)$$

$$\{\gamma_5, \gamma^\mu\} = 0 \quad (105)$$

$$[\gamma_5, \sigma^{\mu\nu}] = 0 \quad (106)$$

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\sigma^{\mu\nu} \quad (107)$$

$$\gamma^\mu\gamma_\mu = 4 \quad (108)$$

$$\gamma^\mu\gamma_5\gamma_\mu = -4\gamma_5 \quad (109)$$

$$\gamma^\mu\gamma^\nu\gamma^\lambda = \eta^{\mu\nu}\gamma^\lambda + \eta^{\nu\lambda}\gamma^\mu - \eta^{\mu\lambda}\gamma^\nu - i\epsilon^{\mu\nu\lambda\sigma}\gamma_5\gamma_\sigma \quad (110)$$

$$\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu \quad (111)$$

$$\gamma^\nu\gamma_5\gamma^\mu\gamma_\nu = 2\gamma_5\gamma^\mu \quad (112)$$

$$\begin{aligned} \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\epsilon &= \eta^{\mu\nu}\eta^{\sigma\epsilon} + \eta^{\nu\sigma}\eta^{\mu\epsilon} - \eta^{\mu\sigma}\eta^{\nu\epsilon} - i\eta^{\mu\nu}\sigma^{\sigma\epsilon} - i\eta^{\nu\sigma}\sigma^{\mu\epsilon} + i\eta^{\mu\sigma}\sigma^{\nu\epsilon} + i\eta^{\mu\epsilon}\sigma^{\sigma\nu} \\ &\quad + i\eta^{\nu\epsilon}\sigma^{\mu\sigma} + i\eta^{\sigma\epsilon}\sigma^{\nu\mu} - i\epsilon^{\mu\nu\sigma\epsilon}\gamma_5 \end{aligned} \quad (113)$$

$$\gamma^\epsilon\sigma^{\mu\nu} = i\eta^{\epsilon\mu}\gamma^\nu - i\eta^{\epsilon\nu}\gamma^\mu + \epsilon^{\mu\nu\epsilon\sigma}\gamma_5\gamma_\sigma \quad (114)$$

$$\sigma^{\mu\nu}\gamma^\epsilon = i\eta^{\nu\epsilon}\gamma^\mu - i\eta^{\mu\epsilon}\gamma^\nu + \epsilon^{\mu\nu\epsilon\sigma}\gamma_5\gamma_\sigma \quad (115)$$

$$\gamma^\mu\sigma^{\sigma\epsilon}\gamma^\nu = i\eta^{\epsilon\nu}\eta^{\mu\sigma} - i\eta^{\sigma\nu}\eta^{\mu\epsilon} + \eta^{\epsilon\nu}\sigma^{\mu\sigma} - \eta^{\sigma\nu}\sigma^{\mu\epsilon} - \epsilon^{\sigma\epsilon\nu\mu}\gamma_5 + i\epsilon^{\sigma\epsilon\nu\lambda}\gamma_5\sigma^\mu{}_\lambda \quad (116)$$

$$\gamma^\sigma\sigma^{\mu\nu}\gamma_\sigma = 0 \quad (117)$$

$$\sigma^{\mu\nu}\gamma_\mu = -3i\gamma^\nu, \quad (118)$$

$$\sigma^{\mu\nu}\gamma^\rho\gamma_\mu = 3i\eta^{\nu\rho} + \sigma^{\nu\rho}, \quad (119)$$

$$\sigma^{\mu\nu}\sigma^{\rho\tau}\gamma_\mu = \eta^{\nu\rho}\gamma^\tau - \eta^{\nu\tau}\gamma^\rho + i\epsilon^{\nu\rho\tau\sigma}\gamma_5\gamma_\sigma, \quad (120)$$

$$\gamma^\mu\sigma_{\nu\mu} = -3i\gamma_\nu, \quad (121)$$

$$\gamma^\mu\gamma^\rho\sigma_{\nu\mu} = 3i\delta_\nu{}^\rho - \sigma_\nu{}^\rho, \quad (122)$$

$$\gamma^\mu\sigma^{\rho\tau}\sigma_{\nu\mu} = \delta_\nu{}^\tau\gamma^\rho - \delta_\nu{}^\rho\gamma^\tau + i\eta_{\nu\kappa}\epsilon^{\kappa\rho\tau\sigma}\gamma_5\gamma_\sigma, \quad (123)$$

$$\begin{aligned} -\epsilon^{\lambda\rho\sigma\epsilon}\epsilon_\lambda{}^{\mu\nu\tau} &= \eta^{\rho\mu}\eta^{\sigma\nu}\eta^{\epsilon\tau} - \eta^{\rho\mu}\eta^{\epsilon\nu}\eta^{\sigma\tau} + \eta^{\rho\nu}\eta^{\sigma\tau}\eta^{\epsilon\mu} - \eta^{\rho\nu}\eta^{\epsilon\tau}\eta^{\sigma\mu} + \eta^{\rho\tau}\eta^{\sigma\mu}\eta^{\epsilon\nu} \\ &\quad - \eta^{\rho\tau}\eta^{\epsilon\mu}\eta^{\sigma\nu} \end{aligned} \quad (124)$$

B Derivation of the Belinfante Fierz identity

Here we supplement section 2.3 with a more detailed version of the derivation of (28). The four Fierz expansions containing the term we want to solve for, $[(\partial_\mu\bar{\psi})\gamma_\nu\psi - \bar{\psi}\gamma_\nu(\partial_\mu\psi)]$ are

$$\begin{aligned} j_\nu[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] &= \frac{i}{3}(\partial_\mu j^\sigma)s_{\nu\sigma} - \frac{i}{3}j^\sigma(\partial_\mu s_{\nu\sigma}) + \frac{1}{3}(\partial_\mu\omega)k_\nu - \frac{1}{3}\omega(\partial_\mu k_\nu) \\ &\quad + \frac{1}{3}\sigma[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] - \frac{i}{3}s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] \\ &\quad - \frac{i}{3}k^\sigma[\bar{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\sigma_{\nu\sigma}\psi], \end{aligned} \quad (125)$$

$$j_\nu[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi] = \frac{i}{3}(\partial_\mu j^\sigma)^*s_{\nu\sigma} - \frac{i}{3}j^\sigma(\partial_\mu^*s_{\nu\sigma}) + \frac{1}{3}(\partial_\mu\sigma)k_\nu - \frac{1}{3}\sigma(\partial_\mu k_\nu)$$

$$\begin{aligned}
& + \frac{1}{3}\omega[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] - \frac{i}{3}s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] \\
& - \frac{i}{3}k^\sigma[\bar{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\sigma_{\nu\sigma}\psi],
\end{aligned} \tag{126}$$

$$\begin{aligned}
k_\nu(\partial_\mu\sigma) &= \frac{1}{3}\sigma(\partial_\mu k_\nu) - \frac{i}{3}\partial_\mu(j^\sigma s_{\nu\sigma}) + \frac{1}{3}j_\nu[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi] \\
& + \frac{i}{3}s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] - \frac{i}{3}k^\sigma[\bar{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\sigma_{\nu\sigma}\psi] \\
& - \frac{1}{3}\omega[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi],
\end{aligned} \tag{127}$$

$$\begin{aligned}
k_\nu(\partial_\mu\omega) &= \frac{1}{3}\omega(\partial_\mu k_\nu) - \frac{i}{3}\partial_\mu(j^\sigma s_{\nu\sigma}) + \frac{1}{3}j_\nu[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] \\
& + \frac{i}{3}s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] - \frac{i}{3}k^\sigma[\bar{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\sigma_{\nu\sigma}\psi] \\
& - \frac{1}{3}\sigma[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi].
\end{aligned} \tag{128}$$

Combining these equations gives

$$\begin{aligned}
[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] &= (\sigma\omega)^{-1} \left(-\frac{i}{2}(\partial_\mu j^\sigma)(\omega s_{\nu\sigma} + \sigma^* s_{\nu\sigma}) - k_\nu[\sigma(\partial_\mu\sigma) + \omega(\partial_\mu\omega)] \right. \\
& + \frac{1}{2}(\partial_\mu k_\nu)(\sigma^2 + \omega^2) + j_\nu\{\omega[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] + \sigma[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi]\} \\
& \left. + \frac{i}{2}(\sigma s_{\nu\sigma} + \omega^* s_{\nu\sigma})[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] \right),
\end{aligned} \tag{129}$$

which obviously requires more Fierz manipulation, since there are still spinor terms present. Using the Dirac identities (118)-(120), we obtain the additional Fierz expansions

$$\begin{aligned}
s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] &= \frac{3i}{5}\sigma(\partial_\mu k_\nu) - \frac{3i}{5}(\partial_\mu\sigma)k_\nu + \frac{1}{5}j^\sigma(\partial_\mu^* s_{\nu\sigma}) - \frac{1}{5}(\partial_\mu j^\sigma)^* s_{\nu\sigma} + \frac{3i}{5}j_\nu[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi] \\
& - \frac{1}{5}k^\sigma[\bar{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\sigma_{\nu\sigma}\psi] + \frac{3i}{5}\omega[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi],
\end{aligned} \tag{130}$$

$$\begin{aligned}
^*s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] &= \frac{3i}{5}\omega(\partial_\mu k_\nu) - \frac{3i}{5}(\partial_\mu\omega)k_\nu + \frac{1}{5}j^\sigma(\partial_\mu s_{\nu\sigma}) - \frac{1}{5}(\partial_\mu j^\sigma)s_{\nu\sigma} + \frac{3i}{5}j_\nu[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] \\
& - \frac{1}{5}k^\sigma[\bar{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\sigma_{\nu\sigma}\psi] + \frac{3i}{5}\sigma[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi].
\end{aligned} \tag{131}$$

Combining these expansions into the form they appear in (129), we get

$$\begin{aligned}
\frac{i}{2}(\sigma s_{\nu\sigma} + \omega^* s_{\nu\sigma})[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] &= \frac{3}{10}k_\nu[\sigma(\partial_\mu\sigma) + \omega(\partial_\mu\omega)] \\
& - \frac{3}{10}(\sigma^2 + \omega^2)(\partial_\mu k_\nu) - \frac{i}{10}(\partial_\mu j^\sigma)(\sigma^* s_{\nu\sigma} + \omega s_{\nu\sigma}) + \frac{i}{10}j^\sigma[\sigma(\partial_\mu^* s_{\nu\sigma}) + \omega(\partial_\mu s_{\nu\sigma})] \\
& + \frac{3}{10}j_\nu(\sigma^2 - \omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2 + \omega^2) + 2im^\sigma(\partial_\mu n_\sigma)\sigma\omega] \\
& - \frac{i}{10}k^\sigma\{\sigma[\bar{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\sigma_{\nu\sigma}\psi] + \omega[\bar{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\sigma_{\nu\sigma}\psi]\}
\end{aligned}$$

$$-\frac{3}{5}\sigma\omega[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi], \quad (132)$$

which itself contains terms requiring further Fierz analysis. Using the Dirac identities (121)-(123), we find that the expansion of these terms is

$$\begin{aligned} k^\sigma[\bar{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\sigma_{\nu\sigma}\psi] &= \frac{1}{5}(\partial_\mu j^\sigma)^*s_{\nu\sigma} - \frac{1}{5}j^\sigma(\partial_\mu^*s_{\nu\sigma}) + \frac{3i}{5}(\partial_\mu\sigma)k_\nu - \frac{3i}{5}\sigma(\partial_\mu k_\nu) \\ &+ \frac{3i}{5}j_\nu[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi] - \frac{1}{5}s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] \\ &+ \frac{3i}{5}\omega[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi], \end{aligned} \quad (133)$$

$$\begin{aligned} k^\sigma[\bar{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\sigma_{\nu\sigma}\psi] &= \frac{1}{5}(\partial_\mu j^\sigma)s_{\nu\sigma} - \frac{1}{5}j^\sigma(\partial_\mu s_{\nu\sigma}) + \frac{3i}{5}(\partial_\mu\omega)k_\nu - \frac{3i}{5}\omega(\partial_\mu k_\nu) \\ &+ \frac{3i}{5}j_\nu[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] - \frac{1}{5}^*s_{\nu\sigma}[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] \\ &+ \frac{3i}{5}\sigma[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi]. \end{aligned} \quad (134)$$

Again, combining these terms into the form in which they appear in (132) gives

$$\begin{aligned} &-\frac{i}{10}k^\sigma\{\sigma[\bar{\psi}\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\sigma_{\nu\sigma}\psi] + \omega[\bar{\psi}\gamma_5\sigma_{\nu\sigma}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\sigma_{\nu\sigma}\psi]\} \\ &= \frac{i}{50}(\sigma s_{\nu\sigma} + \omega^*s_{\nu\sigma})[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] + \frac{6}{50}\sigma\omega[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] \\ &\quad - \frac{3}{50}j_\nu(\sigma^2 - \omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2 + \omega^2) + 2im^\sigma(\partial_\mu n_\sigma)\sigma\omega] - \frac{3}{50}(\sigma^2 + \omega^2)(\partial_\mu k_\nu) \\ &\quad + \frac{3}{50}k_\nu[\sigma(\partial_\mu\sigma) + \omega(\partial_\mu\omega)] + \frac{i}{50}j^\sigma[\sigma(\partial_\mu^*s_{\nu\sigma}) + \omega(\partial_\mu s_{\nu\sigma})] - \frac{i}{50}(\partial_\mu j^\sigma)(\sigma^*s_{\nu\sigma} + \omega s_{\nu\sigma}), \end{aligned} \quad (135)$$

which when substituting into (132) and rearranging, gives

$$\begin{aligned} &\frac{i}{2}(\sigma s_{\nu\sigma} + \omega^*s_{\nu\sigma})[\bar{\psi}\gamma_5\gamma^\sigma(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\gamma^\sigma\psi] = -\frac{1}{2}\sigma\omega[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] \\ &\quad + \frac{1}{4}j_\nu(\sigma^2 - \omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2 + \omega^2) + 2im^\sigma(\partial_\mu n_\sigma)\sigma\omega] - \frac{3}{8}(\sigma^2 + \omega^2)(\partial_\mu k_\nu) \\ &\quad + \frac{3}{8}k_\nu[\sigma(\partial_\mu\sigma) + \omega(\partial_\mu\omega)] - \frac{i}{8}(\partial_\mu j^\sigma)(\sigma^*s_{\nu\sigma} + \omega s_{\nu\sigma}) + \frac{i}{8}j^\sigma[\sigma(\partial_\mu^*s_{\nu\sigma}) + \omega(\partial_\mu s_{\nu\sigma})], \end{aligned} \quad (136)$$

a pure bilinear tensor expression. Now, using the Fierz identities derived in [12]

$$[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] = -(\sigma^2 - \omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\omega + im^\nu(\partial_\mu n_\nu)\sigma], \quad (137)$$

$$[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi] = -(\sigma^2 - \omega^2)^{-1}[j^\nu(\partial_\mu k_\nu)\sigma + im^\nu(\partial_\mu n_\nu)\omega], \quad (138)$$

and combining them into the form in which they appear in (129), we get

$$\begin{aligned} &\omega[\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] + \sigma[\bar{\psi}\gamma_5(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_5\psi] \\ &= -(\sigma^2 - \omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2 + \omega^2) + 2im^\sigma(\partial_\mu n_\sigma)\sigma\omega], \end{aligned} \quad (139)$$

which along with (136), can be substituted into (129) to give

$$[\bar{\psi}\gamma_\nu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma_\nu\psi] = (\sigma\omega)^{-1}\left\{-\frac{1}{2}j_\nu(\sigma^2 - \omega^2)^{-1}[j^\sigma(\partial_\mu k_\sigma)(\sigma^2 + \omega^2) + 2im^\sigma(\partial_\mu n_\sigma)\sigma\omega]\right.$$

$$\begin{aligned}
& + \frac{1}{12}(\sigma^2 + \omega^2)(\partial_\mu k_\nu) - \frac{5}{12}k_\nu[\sigma(\partial_\mu \sigma) + \omega(\partial_\mu \omega)] + \frac{i}{12}j^\sigma[\sigma(\partial_\mu {}^*s_{\nu\sigma}) + \omega(\partial_\mu s_{\nu\sigma})] \\
& - \frac{5i}{12}(\partial_\mu j^\sigma)(\sigma {}^*s_{\nu\sigma} + \omega s_{\nu\sigma}) \Big\}. \tag{140}
\end{aligned}$$

This expression contains no explicit spinor terms, as required, but we can improve it by eliminating the rank-2 tensors $s_{\mu\nu}$ and ${}^*s_{\mu\nu}$, by using the Fierz identity

$$s^{\mu\nu} = (\sigma^2 - \omega^2)^{-1}(\sigma\epsilon^{\mu\nu\rho\sigma} - \omega\delta^{\mu\nu\rho\sigma})j_\rho k_\sigma, \tag{141}$$

$${}^*s^{\mu\nu} = (\sigma^2 - \omega^2)^{-1}(\omega\epsilon^{\mu\nu\rho\sigma} - \sigma\delta^{\mu\nu\rho\sigma})j_\rho k_\sigma, \tag{142}$$

where we define the partially antisymmetric object

$$\delta^{\mu\nu\rho\sigma} \equiv i(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}). \tag{143}$$

We also require the derivatives of these identities, which are

$$\begin{aligned}
\partial_\mu s_{\nu\sigma} &= \frac{\{[2\sigma\omega(\partial_\mu \omega) - (\sigma^2 + \omega^2)(\partial_\mu \sigma)]\epsilon_{\nu\sigma\rho\epsilon} + [2\sigma\omega(\partial_\mu \sigma) - (\sigma^2 + \omega^2)(\partial_\mu \omega)]\delta_{\nu\sigma\rho\epsilon}\}j^\rho k^\epsilon}{(\sigma^2 - \omega^2)^2} \\
& + \frac{(\sigma\epsilon_{\nu\sigma\rho\epsilon} - \omega\delta_{\nu\sigma\rho\epsilon})[(\partial_\mu j^\rho)k^\epsilon + j^\rho(\partial_\mu k^\epsilon)]}{\sigma^2 - \omega^2}, \tag{144}
\end{aligned}$$

$$\begin{aligned}
\partial_\mu {}^*s_{\nu\sigma} &= \frac{\{[-2\sigma\omega(\partial_\mu \sigma) + (\sigma^2 + \omega^2)(\partial_\mu \omega)]\epsilon_{\nu\sigma\rho\epsilon} + [-2\sigma\omega(\partial_\mu \omega) + (\sigma^2 + \omega^2)(\partial_\mu \sigma)]\delta_{\nu\sigma\rho\epsilon}\}j^\rho k^\epsilon}{(\sigma^2 - \omega^2)^2} \\
& + \frac{(\omega\epsilon_{\nu\sigma\rho\epsilon} - \sigma\delta_{\nu\sigma\rho\epsilon})[(\partial_\mu j^\rho)k^\epsilon + j^\rho(\partial_\mu k^\epsilon)]}{\sigma^2 - \omega^2}. \tag{145}
\end{aligned}$$

the rank-2 dependent terms in (140) become

$$\begin{aligned}
\frac{i}{12}j^\sigma[\sigma(\partial_\mu {}^*s_{\nu\sigma}) + \omega(\partial_\mu s_{\nu\sigma})] &= \frac{1}{12}(\sigma^2 - \omega^2)^{-1}\{2\sigma\omega k_\nu[\omega(\partial_\mu \sigma) - \sigma(\partial_\mu \omega)] \\
& + 2i\sigma\omega\epsilon_{\nu\sigma\rho\epsilon}j^\sigma(\partial_\mu j^\rho)k^\epsilon + j_\nu j^\sigma(\partial_\mu k_\sigma)(\sigma^2 + \omega^2)\} - \frac{1}{12}(\partial_\mu k_\nu)(\sigma^2 + \omega^2), \tag{146}
\end{aligned}$$

$$\begin{aligned}
- \frac{5i}{12}(\partial_\mu j^\sigma)(\sigma {}^*s_{\nu\sigma} + \omega s_{\nu\sigma}) &= (\sigma^2 - \omega^2)^{-1}\left\{\frac{5}{12}j_\nu j^\sigma(\partial_\mu k_\sigma)(\sigma^2 + \omega^2) - \frac{5i}{6}\sigma\omega\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma)j^\rho k^\epsilon \right. \\
& \left. - \frac{5}{12}k_\nu(\sigma^2 + \omega^2)[\omega(\partial_\mu \omega) - \sigma(\partial_\mu \sigma)]\right\}, \tag{147}
\end{aligned}$$

giving us the final form of our identity

$$\begin{aligned}
[\bar{\psi}\gamma_\nu(\partial_\mu \psi) - (\partial_\mu \bar{\psi})\gamma_\nu \psi] &= (\sigma^2 - \omega^2)^{-1}\{k_\nu[\omega(\partial_\mu \sigma) - \sigma(\partial_\mu \omega)] - i\epsilon_{\nu\sigma\rho\epsilon}(\partial_\mu j^\sigma)j^\rho k^\epsilon \\
& - ij_\nu m^\sigma(\partial_\mu n_\sigma)\}. \tag{148}
\end{aligned}$$

References

- [1] F. J. Belinfante. On the spin angular momentum of mesons. *Physica*, 6:887, 1939.
- [2] H. S. Booth, G. Legg, and P. D. Jarvis. Algebraic solution for the vector potential in the dirac equation. *J. Phys. A: Math. Gen.*, 34:5667, 2001.
- [3] C. G. Callan, S. Coleman, and R. Jackiw. A new improved energy-momentum tensor. *Ann. Phys.*, 59:42, 1970.

- [4] J. P. Crawford. On the algebra of dirac bispinor densities: Factorization and inversion theorems. *J. Math. Phys.*, 26:1439, 1985.
- [5] N. A. Doughty. *Lagrangian interaction*. Addison-Wesley Publishers Ltd., 1990.
- [6] C. J. Eliezer. A consistency condition for electron wave functions. *Proc. Camb. Phil. Soc.*, 54:247, 1958.
- [7] M. Fierz. Zur fermischen theorie des β -zerfalls. *Z. Phys.*, 104:553, 1937.
- [8] G. H. Goedecke. On stress-energy tensors. *J. Math. Phys.*, 15:792, 1974.
- [9] M. J. Gotay and J. E. Marsden. Stress-energy-momentum tensors and the Belinfante-Rosenfeld formula. *Contemp. Math.*, 132:367, 1992.
- [10] W. Greiner. *Relativistic Quantum Mechanics: Wave Equations*. Springer-Verlag Berlin Heidelberg, 1990.
- [11] S. M. Inglis and P. D. Jarvis. Algebraic inversion of the dirac equation for the vector potential in the non-abelian case. *J. Phys. A: Math. Theor.*, 45:465202, 2012.
- [12] S. M. Inglis and P. D. Jarvis. Fierz bilinear formulation of the maxwell-dirac equations and symmetry reductions. *Ann. Phys.*, 348:176, 2014.
- [13] C. Itzykson and J.-B. Zuber. *Quantum Field Theory*. McGraw-Hill International Book Co. New York, 1980.
- [14] F. A. Kaempffer. Spinor electrodynamics as a dynamics of currents. *Phys. Rev. D*, 23:918, 1981.
- [15] J. Kijowski and G. Rudolph. Spinor electrodynamics in terms of gauge-invariant quantities. *Lett. Math. Phys.*, 29:103, 1993.
- [16] J. Kijowski, G. Rudolph, and M. Rudolph. Functional integral of qed in terms of gauge-invariant quantities. *Lett. Math. Phys.*, 33:139, 1995.
- [17] B. A. Lord. A theorem on stress-energy tensors. *J. Math. Phys.*, 17:37, 1976.
- [18] J. Patera, P. Winternitz, and H. Zassenhaus. Continuous subgroups of the fundamental groups of physics. i. general method and the poincaré group. *J. Math. Phys.*, 16:1597, 1975.
- [19] L. Rosenfeld. Sur le tenseur d'impulsion-énergie. *Acad. Roy. Belg. Memoirs de classes de Science*, 18, 1940.
- [20] T. Takabayasi. Relativistic hydrodynamics of the dirac matter. *Prog. Theor. Phys. Supplement*, 4:1, 1957.
- [21] Y. Takahashi. The fierz identities - a passage between spinors and tensors. *J. Math. Phys.*, 24:1783, 1983.
- [22] R. M. Wald. *General Relativity*. The University of Chicago Press, 1984.
- [23] S. Weinberg. *The Quantum Theory of Fields: Volume I Foundations*. Press Syndicate of the University of Cambridge, 1995.
- [24] J. Yopez. Einstein's vierbein field theory of curved space. *arXiv:1106.2037*, 2011.